

Lorentzian dynamics in the Ashtekar gravity

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We examine the advantages of the $SO(3)$ -ADM (Ashtekar) formulation of general relativity, from the point of following the dynamics of the Lorentzian spacetime in direction of applying this into numerical relativity. We describe our strategy how to treat new constraints and reality conditions, together with a proposal of new variables. We show an example of passing a degenerate point in flat spacetime numerically by posing ‘reality recovering’ conditions on spacetime. We also discuss some available advantages in numerical relativity.

1 Introduction

A decade has passed since the proposal of the new formulation of general relativity by Ashtekar ¹. By using the special pair of variables, the framework has many advantages. That is, the constraint equations which appear in the theory become low-order polynomials and do not contain the inverses of the variables, which enables us to treat the degenerate points. The theory also has the correct form for gauge theoretical features, and suggests possibilities for treating a quantum description of gravity nonperturbatively.

We examine these advantages from the point of numerical relativity. Salisbury *et. al.* ² showed a set of equations using Capovilla-Dell-Jacobson (CDJ) ³ version of connection formulation for vacuum plane symmetric spacetime for numerical demonstration. However, their treatment of reality conditions and slicing condition is not general enough, and not clear what advantages can we get in numerical treatment. Here, we introduce our strategy how to treat new constraints and reality conditions ⁴ and our study of dynamical treatment of a degenerate point ⁵.

2 Ashtekar formulation

The key feature of Ashtekar’s formulation is the introduction of a self-dual connection as one of the basic dynamical variables. Let us write^{*} the metric $g_{\mu\nu}$ using the tetrad, e_μ^I , and define its inverse, E_I^μ , by $g_{\mu\nu} = e_\mu^I e_\nu^J \eta_{IJ}$ and $E_I^\mu := e_\nu^J g^{\mu\nu} \eta_{IJ}$. We define a $SO(3, C)$ self-dual connection $\mathcal{A}_\mu^a := \omega_\mu^{0a} - (i/2) \epsilon_{bc}^a \omega_\mu^{bc}$, where ω_μ^{IJ} is a spin connection 1-form (Ricci connection), $\omega_\mu^{IJ} := E^{I\nu} \nabla_\mu e_\nu^J$. The lapse function, N , and shift vector, N^i , are expressed as $E_0^\mu = (1/N, -N^i/N)$.

Ashtekar treated the set $(\mathcal{A}_a^i, \tilde{E}_a^i)$ as basic dynamical variables, where \tilde{E}_a^i is an inverse of the densitized triad defined by $\tilde{E}_a^i := e E_a^i$, and where $e := \det e_i^a$ is a density. Then the full set of equations are dynamical equations for \mathcal{A}_i^a and \tilde{E}_a^i , together with Hamiltonian constraint \mathcal{C}_H , momentum constraint \mathcal{C}_M and gauge (Gauss) constraint \mathcal{C}_G . Details are in ¹ or in ⁵ with cosmological constant.

^{*} We use $\mu, \nu = 0, \dots, 3$ and $i, j = 1, \dots, 3$ as spacetime indexes, while $I, J = (0), \dots, (3)$ and $a, b = (1), \dots, (3)$ are $SO(1, 3)$, $SO(3)$ indexes respectively.

3 Reality conditions and additional constraint equation

If we compare this formulation with conventional Arnowitt-Deser-Misner (ADM) 3+1 formulation, the bottleneck is additional constraint \mathcal{C}_G and the reality conditions. The reality conditions are, so far, posed on the metric or the triad. We clarified these differences and showed that the triad reality condition restricts three more freedom than the metric reality condition, which fixes the real part of the gauge function \mathcal{A}_0^a (we named this function *triad lapse*)⁴.

Since we are only interested in the dynamics of real Lorentzian spacetime, we have to impose metric reality condition. From the fact that the reality of the spacetime is conserved if we solve reality conditions initially⁶, so we propose to prepare ADM initial data for evolution in Ashtekar's variables by transforming variables and introducing internal variables as they satisfy \mathcal{C}_G .

CDJ solved \mathcal{C}_H and \mathcal{C}_M by introducing new variables, which corresponds to the Weyl curvature Ψ_i . In contrast to CDJ, we make an alternative treatment of the gauge constraint \mathcal{C}_G and the secondary metric reality condition. We summarize these properties in Table 1. Details are in⁴.

	ADM	Ashtekar		CDJ ³	YS ⁴
dynamical variables	γ_{ij}, K_{ij}	$\mathcal{A}_i^a, \tilde{E}_a^i$		$\mathcal{A}_i^a, \Psi_{ab}$	$\Re[\mathcal{A}^{(ab)}], \tilde{E}_a^i$
gauge fixing variables	N, N^i	N, N^i, \mathcal{A}_0^a		N, N^i, \mathcal{A}_0^a	N, N^i, \mathcal{A}_0^a
constraints	$\mathcal{C}_H, \mathcal{C}_M$	$\mathcal{C}_H, \mathcal{C}_M, \mathcal{C}_G$		\mathcal{C}_G	$\mathcal{C}_H, \mathcal{C}_M$
reality conditions	(none)	metric primary 2nd	triad primary 2nd	metric primary 2nd	triad primary metric 2nd (solved)

Table 1: A list of alternative approaches for time evolution of the three-hypersurfaces.

4 Trick for passing a degenerate point

Next, we examine the possibilities of passing a degenerate point. A ‘degenerate point’, we use here, is defined as the point in the spacetime where the density e of 3-space vanishes. In the Ashtekar formulation, all the equations do not include any inverse of e apparently, so that we expect we can ‘pass’ such a degenerate point.

In order to say ‘pass’ degenerate points, we start from requiring the finiteness of the fundamental variables (and their derivatives), $\tilde{E}_a^i, \mathcal{A}_i^a, N/e, N^i, \mathcal{A}_0^a$, and the condition that the calculation must be finished in finite coordinate time. Although these are natural conditions for pursuing the evolutions of spacetime, we concluded that continuing evolutions including a degenerate point in its foliation of 3-space is generally break one of above conditions. The difficulties are that the term ω_i^{bc} in \mathcal{A}_i^a diverges generally and a requirement of finite coordinate time fails when we pass a

degenerate point. This means generally we face a trouble when we pass a degenerate point directly in Lorentzian spacetime even if we use Ashtekar's variables.

However, since the variables are originally defined as complex numbers, if we are allowed to break the reality condition locally in the neighbour of a degenerate point, which we also assume its degeneracy exists only on the real section of spacetime, then we can 'pass' a degenerate point by such a 'deformed slice approach'. Note that, in our proposal, the foliation maintains $3 + 1$ dimensions $\mathbf{R}^3 \times \mathbf{R}$ in \mathbf{C}^4 .

In order to recover a real metric spacetime again later, we have to impose 'reality recovering condition' on the foliation, which requires us to determine shooting parameters in complex part of gauge variables. We showed this technique actually works, by demonstrating a numerical evolution for an analytic solution of degenerate point in flat spacetime⁵. We see that the time evolution does work properly in the sense that the real part of evolution recovers the analytic evolutions and the imaginary part of metric vanishes asymptotically.

5 Discussion

In summary, when we apply Ashtekar's connection approach to Lorentzian dynamics, especially in numerical treatment, expected difficulties such as treatment of reality conditions and additional constraints are conquered by choosing alternative variables and by preparing ADM initial data. One expected advantage of tractability of a degenerate metric requires us to break reality condition locally, but we found a trick to do so.

In the last, we comment on another expected advantages in the applications for numerical relativity: new available slicing conditions. Since we have the additional gauge freedom of 'triad lapse' \mathcal{A}_0^a , there are wide varieties in choosing a slicing condition, if we define it using connection variables. (Note that just rewriting a slicing condition defined in ADM in terms of connection variables has no practical advantages.) For example, Ashtekar variable has close relation with connection or curvature quantities such as Newman-Penrose variables, we expect that we can control curvature or shear locally more directly than ADM variables. A detailed discussion together with numerical demonstrations will be reported elsewhere. We expect that this connection approach to numerical relativity will enable us to study also the dynamics of the signature changing process, topology changing process and causal structure in a complex manifold.

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